QUANTUM PHYSICS I - Oct. 29, 2020

Write your name and student number on **all** sheets. There are three problems in this two-hour exam. You can earn 90 points in total.

PROBLEM 1: QM in 1D (5+10+5+10+10 = 40 points)

Consider a particle in one dimension with a wavefunction $\Psi(x,t)$.

a) For a free particle with $V(x) = 0$, what is the most general form of a stationary state that satisfies the Schrödinger equation? You do not have to incorporate time dependence nor normalizability for the moment.

b) What is the most general one-particle wavefunction that one can construct out of these? Include the time-dependence and explain the normalization condition.

c) Now consider a potential energy with the profile

$$V(x) = \begin{cases} \infty & x < 0\,, \\ -V_0 & 0 < x < x_0\,, \\ 0 & x > x_0\,, \end{cases} \tag{1}$$

where V_0 and x_0 are positive constants. Focus on a possible bound state, i.e. with energy between $-V_0$ and 0. What is the general form of the stationary state in the three intervals ($x < 0$, $0 < x < x_0$ and $x > x_0$)? You do not have to incorporate normalizability and boundary conditions for the moment.

d) What are the boundary conditions and normalizability constraint that must be imposed for a bound state? Incorporate these constraints in your general answer of the previous question. You don't have to solve these constraints at this moment.

e) For a large potential well that is deep (large V_0) and wide (large x_0), there will be many bound states. What happens instead for a narrow potential well, with x_0 small? Use Taylor expansions for small x_0 and only keep the lowest non-trivial order to simplify the boundary conditions. Is there always a bound state, or are there zero bound states for a sufficiently small well? Briefly explain your answer.

PROBLEM 2: CONCEPTS (5+5+5 = 15 points)

a) Prove that, for a Hermitian operator in a finite-dimensional vector space with an inner product, the eigenvalues are always real.

b) Does this property always hold for Hermitian operators in an infinite-dimensional vector space with an inner product? Distinguish between the cases with a discrete and continuous spectrum of eigenvalues. You don't have to perform any calculations, just briefly state whether this property holds in such cases.

c) A multi-particle system can be in an entangled state. Consider two electrons and only focus on their spin orientations. Is the wavefunction

$$\psi = \frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>)\,, \tag{2}$$

entangled or not? Briefly explain / calculate your answer.

PROBLEM 3: QM in 3D (10+5+5+10+5 = 35 points)

Consider a particle in three dimensions with a wavefunction $\Psi(x, y, z, t)$. An important difference with 1D is the possibility of having angular momentum $\vec{L}$. One can construct eigenstates of the compatible operators L^2 and L_z.

a) Consider the linear combinations $L_\pm = L_x \pm iL_y$. Calculate the commutators $[L_\pm, L_z]$ and $[L_\pm, L^2]$. (See formula sheet for $[L_x, L_y]$ and use that $[L_x, L_y^2] = [L_x, L_y]L_y - L_y[L_y, L_x]$ etc.)

b) Explain why these linear combinations are referred to as ladder operators (you do not have to prove any statements here).

c) When written in spherical coordinates, the ladder operators read

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm \frac{i}{\tan(\theta)}\frac{\partial}{\partial\phi}\right)\,. \tag{3}$$

One of the eigenstates of L^2 and L_z is the spherical harmonic $Y_1^0 = A\cos(\theta)$, with eigenvalues denoted by $l = 1$ and $m = 0$ (and A is a specific constant for normalization). Construct the spherical harmonic Y_1^1 (normalization not important).

d) Explain what will happen when applying the raising operator L_+ twice to the spherical harmonic Y_1^0. Illustrate this explicitly given the above eigenstate.

e) When putting two particles together, one has to consider the addition of their angular momenta. Consider two particles that both have one unit of angular momentum, i.e. $l = 1$. Upon measuring the magnitude of the angular momentum of the total system, what is/are the possible outcome(s) for this quantum number?